SKEW PRODUCTS AND RANDOM WALKS ON INTERVALS

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Abstract. In this paper we consider step skew products over a transitive subshift of finite type (topological Markov chain) with an interval fiber. For an open and dense set of such skew products we give a full description of dynamics. Namely, there exist only finite collection of alternating attractors and repellers; we also give an upper bound for their number. Any of them is a graph of a continuous map from the base to the fiber defined almost everywhere w.r.t. any ergodic Markov measure in the base. The orbits starting between the adjacent attractor and repeller tend to the attractor as $t \to +\infty$, and to the repeller as $t \to -\infty$. The attractors support ergodic hyperbolic SRB measures.

There is a natural way to associate a random dynamical system to a step skew product. We show that any generic random dynamical system of this form has finitely many ergodic stationary measures. Each measure has negative Lyapunov exponent.

1. Introduction

Among the dynamical systems, the open set of hyperbolic ones (also known as Axiom A) is the best understood. For them, Smale's Spectral Decomposition theorem [27] allows one to split the non-wandering set into a finite collection of locally maximal hyperbolic sets. These unit sets admit finite Markov partitions and, therefore, symbolic encodings. If any of them is an attractor, then it also carries an SRB (physical) measure [26], [25], [7]. Thus the hyperbolic systems have both good dynamical and statistical descriptions.

From the works of Abraham, Smale [1] and Newhouse [23] we know that Axiom A is not dense in dim ≥ 2. So the next step to understand generic systems is to consider partially hyperbolic (PH) ones [11]. The skew products over hyperbolic sets provide important examples of PH dynamics. Their central bundle is trivial and tangent to the fibers. What makes it worthy to consider the PH skew products is that by Hirsch–Pugh–Shub [12] theory, the systems *conjugated* to such skew products form an open subset in PH. So such skew products are in a sense locally generic.

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The Markov encoding of the base reduces them to the skew products over subshifts of finite type, see Def. 2.1. The simplest skew products which correspond to $\dim E^c = 1$ have one-dimensional fibers. There is the single compact one-dimensional manifold without boundary, the circle S^1, and the single one with boundary, the interval $I = [0, 1]$. The present work deals with skew products whose fibers are unit intervals. Fiber maps are C^1-diffeomorphisms on the image. In this paper, we focus on an important class of skew products, so-called *step* skew products, see Def. 2.2.

Skew products are also widely used as a tool for the construction of (robust) examples of complicated behavior: convergence of orbits coexisting with minimality [2, 20], non-removable zero Lyapunov exponents [10], attractors with intermingled basins [18, 5, 15], multidimensional robust non-hyperbolic attractors [28], and others. So it is very challenging to obtain a description of their dynamics.

To date, more attention has been paid to skew products with circle fibers than to those with interval fibers. Such dynamical systems turned out to exhibit the effects that cannot be observed for one (generic) circle diffeomorphism. In his paper [2], Antonov proved that in an open set of step skew product (see Def. 2.2) with circle fibers the fiberwise coordinates of almost all orbits with the same base coordinate approach each other. Later in [20], this effect was re-discovered by Kleptsyn and Nalski in the following terms. A generic step skew product with circle fibers may have a measurable "attracting" section, which is fiberwise approached by orbits of almost every point. At the same time the section is dense in the phase space. This construction was generalized to the case of mild (not necessarily step) skew products by Homburg in [13]. Recall that the orbits of a single circle diffeomorphism either are dense (irrational rotation number) or tend to attracting periodic points (rational rotation number). The examples [2], [20], [13] exhibit a mixture of these two types of behavior.

The dynamical properties of generic diffeomorphisms of closed manifolds are known to be different from those of generic dissipative diffeomorphisms (that is, diffeomorphisms onto the image of a compact manifold with boundary) in the same dimension. For instance, a generic diffeomorphism of an interval has a finite number of fixed points. Any other point tends toward a fixed point. On the other hand, minimal diffeomorphisms of a circle are (metrically) generic. This difference in behavior motivates us to go from studying skew products with circle fibers to skew products with interval fibers.

Note that each skew product with interval fibers can be extended to a skew product with circle fibers. Namely, think of each interval fiber as of an arc of a circle and extend fiber maps to be circle diffeomorphisms. The simplest way to do it is to put a repelling fixed point at the complement to the arc. The resulting skew product with circle fibers can be made the same regularity of the fiber maps and the dependency of the fibers on the base as the initial interval skew product. Thus our result also describes the dynamics of an open (not dense) set of circle skew products.

In the main theorem of this paper we show that the dynamics of a generic step skew product with interval fibers can be described in relatively simple terms. This dynamics

is similar to the cartesian product of the dynamics in the base and the dynamics of a single interval diffeomorphism. More precisely (see Thm. 2.15), the phase space of such a skew product can be covered by finitely many absorbing and expelling strips. Any absorbing strip contains a unique attractor; any expelling strip contains a unique repeller. These attractors and repellers are "bony graphs" of maps from the base to the fiber: each of them intersects almost every fiber (w.r.t. the Markov measure in the base) at a single point; other fibers are intersected at intervals. This feature is similar to "porcupine horseshoes" discovered by Díaz and Gelfert in [8]. Almost every point of the phase space (w.r.t. the *standard measure* which is the product of the Markov measure in the base and Lebesgue measure in the fiber, sf. Def. 2.4) tends to one of the attractor graphs. When the time is reversed, almost every point either tends to one of the repeller graphs, or is eventually taken to a domain where the inverse map is not defined.

Finally, the closure of each attractor graph and of each repeller graph is the support of an ergodic invariant measure that projects to the Markov measure in the base. The attractor measures are SRB-measures; their basins of attraction contain subsets of full standard measure in the corresponding strips. The fiberwise Lyapunov exponents of the attracting and the repelling measures are strictly negative and strictly positive, respectively.

In fact, our results are valid not only for the Markov measures but for a wider class of shift-invariant measures. The only condition we really need is stated in Prop. 6.13. Roughly speaking, it says that for any fixed "right tail" $(\omega_1, \dots, \omega_n, \dots)$ of the symbolic sequence in the base, the conditional probability to see any allowed symbol p_j at the position ω_0 is bounded from zero.

Similar statements also hold for generic mild (see Def. 2.2) skew products with interval fibers; however, the genericity conditions must be changed. The proofs for this case require ideas beyond the present work, and we intend to present them in a separate paper.

In his paper [22], Kudryashov has recently constructed a robust example of a step skew product with interval fibers such that its attractor intersects some fibers at intervals (instead of points). The set of such fibers has continuum cardinality (in fact, it has a Cantor subset). Thus it is not possible to prove the stronger statement that the attractors and repellers intersect each fiber at a single point.

A significant part of this work is devoted to random dynamical systems on an interval. Namely, we consider any ergodic Markov process with N states, $N < \infty$, and N diffeomorphisms of an interval which are applied accordingly to the current state of the process. In Thm. 4.7 we prove that a generic random dynamical system of this kind has only finitely many ergodic stationary measures; their random Lyapunov exponents are strictly negative.

In this paper we always assume that the fiberwise maps preserve the orientation of the unit interval. When this is not the case, our results can be derived as follows. Pass to the orientation-preserving covering. Namely, take two copies of the set of Markov

states, marked with + and -. The orientation-preserving fiber maps take + to + and - to -. The orientation-changing fiber maps take + to - and - to +. Then our main result can be applied. After that we project the attractors and repellers back to the initial space.

Outline of the paper. The proof of our main result, Thm. 2.15, is based on the link between the dynamics of skew products over one- and two-sided Markov shifts and the corresponding random dynamics on the interval.

In Sec. 2 and Sec. 3 we recall some standard definitions, introduce the geometrical structures we need later, and state Thm. 2.15. In Sec. 4 we define random dynamics associated with step skew products. At the end of this section, we state the second main result of the paper, Thm. 4.7. This theorem describes the stationary measures of such random dynamics. In these two sections, an experienced reader can skip everything but the statements of the theorems.

In Sec. 5 we give genericity conditions for Thm. 2.15 and Thm. 4.7. Sec. 6 and7 are devoted to the proof of Thm. 4.7: as we have already mentioned, this proof is based on the link between the skew products and the corresponding random dynamics (cf. Lemmata 6.2, 6.3, 6.7, 6.8). Finally, in Sec. 8 we deduce Thm. 2.15 from Thm. 4.7.

2. Definitions and the main statement

Suppose $\sigma\colon \Sigma \to \Sigma$ is a transitive subshift of finite type (a topological Markov chain) with a finite set of states $\{1, \dots, N\}$ and $A = (a_{ij})_{i,j=1}^N$ is the transition matrix of σ, where $a_{ij} \in \{0, 1\}$. Recall that Σ is the set of all bi-infinite sequences $\omega = (\omega_n)_{-\infty}^{+\infty}$ composed of symbols $1, \dots, N$ such that $a_{\omega_n\omega_{n+1}} = 1$ for any $n \in \mathbb{Z}$ (see, for instance, [19]).

The map σ shifts any sequence ω one step to the left: $(\sigma\omega)_n = \omega_{n+1}$. It is easy to show that the transitivity property of a subshift is equivalent to the following condition for the matrix A:

$$\exists n \in \mathbb{N}\ \forall i, j\ (A^n)_{ij} > 0.$$

Transitivity implies the indecomposability of the subshift. Indeed, for any $m > 0$ the subshift σ^m with the same states allows one to go from any state to any other in finitely many steps. Thus for any $m > 0$ the subshift σ^m cannot be split into two nontrivial subshifts of finite type.

We endow Σ with a metric defined by the formula

$$d(\omega^1, \omega^2) = \begin{cases} 2^{-\min\{|n|\,:\,\omega_n^1 \neq \omega_n^2\}}, & \omega^1 \neq \omega^2, \\ 0, & \omega^1 = \omega^2, \end{cases} \qquad \omega^1, \omega^2 \in \Sigma. \tag{2.1}$$

Now let M be a smooth manifold with boundary.

Definition 2.1. A *skew product* over a subshift of finite type (Σ, σ) is a dynamical system $F\colon \Sigma \times M \to \Sigma \times M$ of the form

$$(\omega, x) \mapsto (\sigma\omega, f_\omega(x)),$$

where $\omega \in \Sigma$, $x \in M$, and the map $f_\omega(x)$ is continuous in ω. The phase space of the subshift is called the *base* of the skew product, the manifold M is called the *fiber*, and the maps f_ω are called the *fiber maps. The fiber over* ω is the set $M_\omega := \{\omega\} \times M \subset \Sigma \times M$.

In any argument about the geometry of the skew products we always assume that the base factor of $\Sigma \times M$ is "horizontal" and the fiber factor is "vertical".

Definition 2.2. A skew product over a subshift of finite type is a *step skew product* if the fiber maps f_ω depend only on the position ω_0 in the sequence ω. For the general skew products (fiber maps depend on the whole sequence ω) we sometimes use the word *mild.*

The fiber M is always the unit interval $I = [0, 1]$ and C^1-diffeomorphisms $f_\omega\colon I \to f(I)$ map the interval strictly inside itself preserving the orientation. In particular, all maps f_ω are strictly increasing. Suppose $\mathcal{M}$ is the set of all such skew products and $\mathcal{S} \subset \mathcal{M}$ is the subset of all step skew products. We endow $\mathcal{M}$ with the metric

$$\mathrm{dist}_{\mathcal{M}}(F, G) := \sup_\omega(\mathrm{dist}_{C^1}(f_\omega, g_\omega)).$$

On $\mathcal{S}$ this induces the max-metric of the Cartesian product of N copies of $\mathrm{Diff}^1(I)$ with the C^1-metric. In this paper, we consider only step skew products. We describe the dynamics of a generic $F \in \mathcal{S}$.

In the rest of this text, all measures are assumed to be probabilities. Like any dynamical system on a compact metric space, the subshift σ has a non-empty set of invariant measures. There is a natural class of them called *Markov measures.* They are defined as follows.

Let $\Pi = (\pi_{ij})_{i,j=1}^N$, $\pi_{ij} \in [0, 1]$ be a right stochastic matrix (i.e., $\forall i \sum_j \pi_{ij} = 1$) such that $\pi_{ij} = 0$ whenever $a_{ij} = 0$. Let p be its eigenvector with non-negative components that corresponds to the eigenvalue 1:

$$\forall i\; p_i \geq 0, \text{ and } \sum_i \pi_{ij} p_i = p_j. \tag{2.2}$$

We can always assume $\sum_i p_i = 1$.

For any finite word $\omega_k \ldots \omega_m$, $k, m \in \mathbb{Z}$, $k \leq m$ we consider a *cylinder*

$$C_w := \{\omega' \in \Sigma \mid \omega'_k = \omega_k, \ldots, \omega'_m = \omega_m\}.$$

The cylinders form a countable base of the topology on Σ. Thus a Borel measure on Σ is properly defined by its values on every cylinder.

Definition 2.3. ν is a *Markov measure* constructed from the distribution p_i and the transition probabilities π_{ij} if its values on the cylinders are:

$$\nu(C_w) := p_{\omega_k} \cdot \prod_{i=k}^{m-1} \pi_{\omega_i \omega_{i+1}}. \tag{2.3}$$

It is easy to see that the formula (2.3) is consistent on the set of all cylinders. Thus ν is well-defined. Moreover, it is invariant under the shift map σ. Note that for any stochastic matrix Π there exists at least one vector p satisfying (2.2). Such a vector is unique whenever the subshift is transitive and $\pi_{ij} \neq 0 \Leftrightarrow a_{ij} \neq 0$ (as in our case). If that is so, the measure ν is ergodic; $\operatorname{supp} \nu$ coincides with Σ.

Let ν be any ergodic Markov measure on Σ. From now on, the measure ν is fixed.

Definition 2.4. The *standard measure* $\mathbf{s}$ on $\Sigma \times I$ is the product of ν and the Lebesgue measure on the fiber.

Our goal is to give the description of the behavior of almost every orbit w.r.t. the standard measure. Such a description is given by Thm. 2.15 below. This is the main result of this paper.

Definition 2.5. We say that a closed set in the skew product is a *bony graph* if it intersects almost every fiber (w.r.t. ν) at a single point, and any other fiber at an interval (a “bone”).

The name comes from the following simple observation. Any bony graph can be represented as the disjoint union of sets K and Γ. The set K is the union of the bones. The projection $h(K)$ onto Σ has zero measure. Γ is a graph of some function $\varphi\colon \Sigma \setminus h(K) \to I$.

For any set B, denote $B_\omega := B \cap I_\omega$.

Definition 2.6. A bony graph B is a *continuous-bony graph (CBG)* if B_ω is upper-semicontinuous:

$$\forall \omega\, \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \operatorname{dist}(\omega, \omega') < \delta \Rightarrow B_{\omega'} \subset U_\varepsilon(B_\omega).$$

By Fubini’s Theorem, the standard measure of a measurable-bony graph is zero.

In Thm. 2.15, we show that the attractors and the repellers of generic skew products are CBGs. They are also F-invariant. It is easy to see that the restriction of the dynamics to an invariant CBG is very much the same as the restriction to the base:

Proposition 2.7. *Suppose B is a CBG, the set K and the function φ are from the Definition 2.6. Then the vertical projection $h\colon B \setminus K \to \Sigma \setminus h(K)$ is a homeomorphism conjugating the dynamics of $F|_{B\setminus K}$ and $\sigma|_{\Sigma\setminus h(K)}$.*

The following geometrical notions are crucial for our reasoning.

Definition 2.8. If the following holds for two functions $\varphi_1, \varphi_2 \colon \Sigma \to I$,
$$\forall \omega \in \Sigma \quad \varphi_1(\omega) < \varphi_2(\omega),$$
we write $\varphi_1 < \varphi_2$. We also write $\Gamma_{\varphi_1} < \Gamma_{\varphi_2}$ for their graphs Γ_{φ_1} and Γ_{φ_2}.

This definition admits a natural extension to the case of bony graphs:

Definition 2.9. Let B_1, B_2 be two bony graphs. We write $B_1 < B_2$ whenever for any $(\omega, x_1) \in B_1$ and $(\omega, x_2) \in B_2$ we have $x_1 < x_2$.

Recall that a skew product permutes the fibers. Thus the image $F(\Gamma)$ of any graph Γ is also a graph of some function.

Definition 2.10. We say that a graph Γ *moves up (down)* if $F(\Gamma) > \Gamma$ (respectively, $F(\Gamma) < \Gamma$).

Definition 2.11. Let $\varphi_1, \varphi_2 \colon \Sigma \to I$ be continuous functions such that $\varphi_1 < \varphi_2$. The set
$$S_{\varphi_1,\varphi_2} := \{(\omega, x) \mid \varphi_1(\omega) \le x \le \varphi_2(\omega)\}$$
is called the *strip between the graphs of* φ_1 *and* φ_2.

Definition 2.12. The strip S_{φ_1,φ_2} is *trapping (nonstrictly trapping)* if $F(S_{\varphi_1,\varphi_2}) \subset \operatorname{int} S_{\varphi_1,\varphi_2}$ (respectively $F(S_{\varphi_1,\varphi_2}) \subset S_{\varphi_1,\varphi_2}$). The strip is *inverse trapping* (*nonstrictly inverse trapping*) if the same holds true for F^{-1}.

Remark 2.13. Because the fiber maps are monotonous, $\varphi_1 < \varphi_2$ implies the inequality $F^n(\varphi_1) < F^n(\varphi_2)$ for any $n \ge 0$. This is also true for any $n < 0$, provided that the preimages of the graphs are well-defined. Thus for any $n \ge 0$ the image $F^n(S_{\varphi_1,\varphi_2})$ is also a (non-empty) strip. Therefore any trapping strip has a non-empty *maximal attractor*:
$$A_{\max}(S_{\varphi_1,\varphi_2}) := \bigcap_{n=0}^{+\infty} F^n(S_{\varphi_1,\varphi_2}). \tag{2.4}$$

In Thm. 2.15 we show that the maximal attractor of any trapping strip is a CBG, provided the strip is indecomposable in a certain sense .

Among all invariant measures of a smooth dynamical system, the measures of Sinai-Ruelle-Bowen (*SRB measures*) are of particular interest. See, for instance, the handbook [3]. The following definition dates back to the classical papers [26, 25, 6]:

Definition 2.14. Let $\mathbf{m}$ be an F-invariant measure. Consider the set V of all points $p \in X$ such that for any function $\varphi \in C(\Sigma \times I)$ the time average is equal to the space average:
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(F^i(p)) = \int\limits_{\Sigma\times I} \varphi \, d\mathbf{m}.$$
$\mathbf{m}$ is an *SRB measure* if the set V has a positive standard measure. The set V is then called its *basin*.

It turns out that in our case the following stronger property holds. The basin of any SRB measure is a full-measure subset of an open subset of X.

In general, very few invariant measures are SRB measures. A long-standing open question asks if the attractors of typical dynamical systems carry SRB measures (see, for instance, the review [24]). In his paper [25], Ruelle proved that uniformly hyperbolic attractors do. In our case the attractors are partially hyperbolic instead, but we also manage to prove the existence of SRB measures for them.

The main result of the paper is the following

Theorem 2.15. *For a generic $F \in \mathcal{S}$ there is a finite collection of trapping strips and inverse trapping strips such that*

i) *their union is the whole phase space X;*
ii) *the maximal attractors (2.4) of the trapping strips are CBGs; the repellers (i.e., the maximal attractors for the inverse map) of every inverse trapping strip are also CBGs;*
iii) *any trapping strip and any inverse trapping strip has a unique ergodic invariant measure such that its projection to the base is the Markov measure ν. Namely, it is the lift of ν onto the attractor (or repeller) considered as the graph of a function defined almost everywhere. This measure is SRB. Its basin contains a subset of full measure of the strip;*
iv) *for any trapping strip, there exists a subset Γ of full measure (w.r.t. the measure from iii) of its maximal attractor, and a set $h\Gamma \subset \Sigma$ of full measure ν such that the vertical projection $h\colon \Gamma \to h\Gamma$ is a homeomorphic conjugacy. Note that $h\Gamma$ is not closed but is invariant;*
v) *the fiber-wise Lyapunov exponents of the attractor and repeller measures from iii are non-zero;*
vi) *the graphs of the attractors and repellers are mutually comparable in the sense of Def. 2.8. Moreover, the attractors and the repellers are alternating: $\dots < A_1 < R_1 < A_2 < R_2 < \dots$.*

Here the set of all generic skew products is an open and dense subset of the space of all step skew products. In Sec. 5 we explicitly state the genericity conditions.

We also provide an upper bound for the number of attractors and repellers, see Prop. 8.6.

Remark 2.16. One may consider the following generalization of step skew products. Let the fiber maps f_ω depend on finitely many symbols $(\omega_{-k} \dots \omega_l)$, $k, l \in \mathbb{N}$, rather than on a single symbol. We call such systems *multistep skew products*.

The statements of Thm. 2.15 also hold for generic multistep skew products. Indeed, for any multistep skew product F there exists a step skew product G. The number of Markov states of the base of G is equal to the number of the admissible words of length $k + l + 1$ in the base of F. G is transitive if F is.

3. Skew products over one-sided shifts

Let Σ_+ be the space of one-sided (infinite to the right) sequences $\omega = (\omega_n)_0^{+\infty}$ satisfying $a_{\omega_n\omega_{n+1}} = 1$ for all n. The left shift $\sigma_+ \colon \Sigma_+ \to \Sigma_+$, $(\sigma_+\omega)_n = \omega_{n+1}$ defines a non-invertible dynamical system on Σ_+. The system (Σ_+, σ_+) is a factor of the system (Σ, σ) under the "forgetting the past" map $\pi \colon (\omega_n)_{-\infty}^{+\infty} \mapsto (\omega_n)_0^{+\infty}$:

$$\begin{tikzcd} \Sigma \arrow[r, "\sigma"] \arrow[d, "\pi"'] & \Sigma \arrow[d, "\pi"] \\ \Sigma_+ \arrow[r, "\sigma_+"] & \Sigma_+ \end{tikzcd} \tag{3.1}$$

Formulae (2.1) and (2.3) define the metric and the invariant measure ν_+ on Σ_+. A measure is called *invariant* under a non-invertible map F_+ if for any measurable set A

$$\nu_+(F_+^{-1}(A)) = \nu_+(A).$$

Recall that in a step skew product over the two-sided Markov shift the fiberwise maps depend only on ω_0. Thus one can pass from the skew product to the quotient:

$$\begin{tikzcd} \Sigma \times I \arrow[r, "F"] \arrow[d, "\pi\times\mathrm{Id}"'] & \Sigma \times I \arrow[d, "\pi\times\mathrm{Id}"] \\ \Sigma_+ \times I \arrow[r, "F_+"] & \Sigma_+ \times I, \end{tikzcd} \tag{3.2}$$

where $F_+(\omega, x) = (\sigma_+\omega, f_{\omega_0}(x))$.

There is a relation between the invariant measures of F and F_+.

Proposition 3.1. i) *For any F-invariant measure $\mathbf{m}$ its projection $\mathbf{m}_+ = \pi_*\mathbf{m}$,*

$$\mathbf{m}_+(A) := \mathbf{m}(\pi^{-1}(A)),$$

is F_+-invariant;

ii) *For any F_+-invariant measure $\mathbf{m}_+$ there exists an F-invariant measure $\mathbf{m}$ such that $\mathbf{m}_+ = \pi_*\mathbf{m}$.*

Proof. The first statement follows immediately from (3.2). To prove the second one, take an arbitrary measure $\bar{\mathbf{m}}$ on $\Sigma \times I$ such that $\mathbf{m}_+ = \pi_*\bar{\mathbf{m}}$. Applying the Krylov-Bogolyubov averaging procedure [21] to $\bar{\mathbf{m}}$, we obtain an F-invariant measure $\mathbf{m}$. And as the measure $\mathbf{m}_+$ is F_+-invariant, (3.2) implies that the projection of $\mathbf{m}$ is also equal to $\mathbf{m}_+$. $\square$

4. Stationary measures of random walks

We assign a random dynamical system to any one-sided skew product, as follows. Imagine that we are tracking the F_+-iterations of a point $(\omega, x) \in \Sigma_+ \times I$, where ω is chosen randomly with respect to the measure ν. But we can only observe its I-coordinate

and the symbol ω_0. Then the sequence of our observations is a realization of a discrete Markov process $\Pi(F)$ on the space $\mathcal{I} = \{1, \dots, N\} \times I$. The transition probability from a point (i, x) to a point $(j, f_i(x))$ equals π_{ij}.

For any measure μ on the space $\mathcal{I}$, it is natural to denote its stochastic image $f_*\mu$ as

$$(f_*\mu)_i := \sum_j \pi_{ji} \cdot (f_j)_*\mu_i, \tag{4.1}$$

where μ_i is the restriction of the measure μ to the interval $I_i = \{i\} \times I$.

Definition 4.1. A measure μ on the space $\mathcal{I}$ is *stationary* if $f_*\mu = \mu$.

Any stationary measure of the process $\Pi(F)$ induces an invariant measure of F_+ that projects to the measure ν_+ on the base. Namely, a stationary measure μ corresponds to the measure

$$\mathbf{m}_+(\mu) := \sum_k \nu_k^+ \times \mu_k, \tag{4.2}$$

where ν_k^+ is the conditional Markov measure on the cylinder $C_{+,k} = \{\, \omega \mid \omega_0 = k\} \subset \Sigma_+$ defined by ν_+:

$$\nu_k^+ = \frac{\nu_+|_{C_{+,k}}}{\nu_+(C_{+,k})}.$$

On the other hand, Prop. 3.1 establishes a relation between invariant measures of F_+ and F. In Prop. 8.3 below we show that the "attractor" measures (see claim iii of Thm. 2.15) are projected under π-factorization to the measures of type (4.2) which correspond to the stationary measures of $\Pi(F)$. However, the "repeller" invariant measures are mapped under π-factorization to measures which are supported on CBGs themselves. So, to study these measures, we will pass from F to the inverse skew product F^{-1}.

Definition 4.2. A pair (k, m) is *admissible* if $\pi_{km} \neq 0$. The corresponding maps $f_k \colon I_k \to I_m$ are also called *admissible.*

Definition 4.3. A subset of $\mathcal{I}$ is a *domain* if it intersects each I_k at a nonempty interval.

Definition 4.4. A domain $D = \bigsqcup_k D_k \subset \mathcal{I}$ is *trapping (nonstrictly trapping)* if any admissible map takes it to its interior (respectively, to itself). In other words,

$$\forall k, m: \ \pi_{km} > 0, \ f_k(D_k) \subset \operatorname{int} D_m \quad (\, f_k(D_k) \subset D_m\,).$$

It easy to see that the following proposition holds.

Proposition 4.5. *The following conditions are equivalent:*

i) *the domain $D = \bigsqcup_k D_k \subset \mathcal{I}$ is trapping (nonstrictly trapping);*
ii) *the strip*

$$\tilde{D}_+ = \bigsqcup_k C_{+,k} \times D_k \subset \Sigma_+ \times I$$

is a trapping (nonstrictly trapping) region for the skew product F_+;

iii) *the strip*

$$\tilde{D} = \bigsqcup_k C_k \times D_k \subset \Sigma \times I \tag{4.3}$$

is trapping (nonstrictly trapping) for F.

Here, as before, $C_{+,k} = \{\omega \mid \omega_0 = k\} \subset \Sigma_+$, and $C_k = \{\omega \mid \omega_0 = k\} \subset \Sigma$.

Definition 4.6. A finite collection of domains is *vertically ordered* if they can be enumerated in such a way that for any interval I_k and for any $i < j$ the intersection of I_k with the ith domain is situated below the intersection of I_k with the jth domain.

Finally, we state our main result on the stationary measures of $\Pi(F)$:

Theorem 4.7. *Suppose F is a generic (in the sense of Sec. 5) step skew product. Then the following statements hold for the random process $\Pi(F)$:*

i) *there exist only finitely many ergodic stationary measures;*
ii) *the supports of these measures are contained in disjoint vertically ordered trapping domains;*
iii) *the (random) Lyapunov exponents of these measures are negative.*

5. Genericity conditions

Definition 5.1. An *admissible composition* is a map of the form

$$f_{w_1\ldots w_n} := f_{w_n} \circ \cdots \circ f_{w_1} : I_{w_1} \to I_{w_{n+1}},$$

where any pair of consequent symbols (w_i, w_{i+1}) is admissible. An admissible composition is called a *simple transition* if all the symbols w_i, $i = 1, \ldots, n+1$ are different. An admissible composition is called a *simple return* if $f_{w_1\ldots w_{n-1}}$ is a simple transition and $w_1 = w_{n+1}$.

Now we can explicitly state the genericity conditions for Thm. 2.15 and Thm. 4.7:

i) Any fixed point p of any simple return g is hyperbolic: $g'(p) \neq 1$;
ii) No attracting fixed point of a simple return is mapped to a repelling fixed point of a simple return by a simple transition. Also, no repelling fixed point of a simple return is mapped to an attracting fixed point of a simple return by a simple transition.
iii) One cannot choose from each interval I_k a single point a_k such that for any admissible couple (i, j) one has $f_i(a_i) = a_j$ (in other words, the set $\{a_k\}_{k=1}^N$ is invariant under random dynamics).

Remark 5.2. As there is only a finite number of simple transitions and simple returns, the above genericity conditions are satisfied on the complement to a finite union of codimension one subsets of $\mathcal{S}$.

Remark 5.3. The condition iii is closely related to the *accessibility* property of partially hyperbolic systems (see, for instance, the handbook [3]).

Remark 5.4. Instead of two latter conditions one may impose the following: there is no way to take a fixed point of a simple return to a fixed point of a simple return by a simple transition map, except for the case when the word defining the simple transition is a suffix of the word defining the latter simple return.

Remark 5.5. Generally speaking, Theorems 4.7 and 2.15 are false without these assumptions. For example, Ilyashenko in [14] gave an example of "thick" (positive measure) attractors appearing when the condition ii is broken.

6. Proof of the stationary measures theorem

Let μ be an arbitrary ergodic stationary measure for the random process $\Pi(F)$ described in Sec. 4. Denote by μ_k the restriction $\mu|_{I_k}$. Let $I_{\mu,k} = [A_{\mu,k}, B_{\mu,k}]$ be the interval that spans the support of μ_k:

$$A_{\mu,k} := \min \operatorname{supp} \mu_k, \quad B_{\mu,k} := \max \operatorname{supp} \mu_k.$$

Note that the interval $I_{\mu,k}$ may not coincide with $\operatorname{supp} \mu_k$.

As μ is stationary, for any admissible transition (i, j) we have

$$f_i(\operatorname{supp} \mu_i) \subset \operatorname{supp} \mu_j.$$

Because the maps f_k are monotonous, the disjoint union of the intervals $I_{\mu,k}$ is forward-invariant: for any admissible transition (i, j)

$$f_i(I_{\mu,i}) \subset I_{\mu,j}. \tag{6.1}$$

Thus the domain $\mathcal{I}_\mu = \bigsqcup_k I_{\mu,k}$ is nonstrictly trapping.

Remark 6.1. The genericity conditions of Sec. 5 imply that no interval $I_{\mu,k}$ can be a single point. Otherwise by (6.1) any interval $I_{\mu,k}$ is a single a point. And this contradicts with the genericity condition iii.

Lemma 6.2. *There exist arbitrarily small strictly trapping neighborhoods of $\mathcal{I}_\mu$.*

Before proving Lemma 6.2, we need some auxiliary statements.

Lemma 6.3. *For any k there exist an attracting fixed point A of a simple return and a simple transition f such that $A_{\mu,k} = f(A)$. The same is true for $B_{\mu,k}$.*

Proof. By the Kakutani random ergodic theorem [17] (see also [9]) a generic sequence of random iterations (k_n, x_n), $x_n \in I_{k_n}$ of a μ-generic initial point is distributed with respect to the measure μ. Let us choose and fix such a generic initial point (k_0, x_0), different from (k_0, A_{μ,k_0}) and (k_0, B_{μ,k_0}). Denote by (k_n, x_n) a generic sequence of iterations of (k_0, x_0). The points (k_n, x_n) are distributed w.r.t. μ.

Thus for any k the set $X_k = \{x_n\}|_{k_n=k}$ is dense in $\operatorname{supp} \mu_k$. In particular, the points $(k, A_{\mu,k})$ and $(k, B_{\mu,k})$ are accumulation points of X_k.

Now we modify the sequence (k_n, x_n) so that it becomes monotonous in the following sense:

Definition 6.4. An admissible sequence of iterations is called *downwards monotonous* if for any (k_m, x_m), (k_n, x_n) such that $k_m = k_n$ and $m < n$ we have $x_m > x_n$. In the same way we define *upwards monotonicity.*

The following proposition selects a monotonous subsequence from any finite sequence of iterations.

Proposition 6.5. *For any finite admissible sequence of iterations*

$$(w_0, x_0), \dots, (w_n, x_n)$$

there exists downwards monotonous finite admissible sequence

$$(w_0, x_0) = (w'_0, x'_0), \dots, (w'_{n'}, x'_{n'}),$$

such that $w'_{n'} = w_n$ *and* $x'_{n'} \le x_n$.

Proof. To find such a sequence, let us write out the symbols of the original sequence, and remove each simple return such that the point in fiber after that return is higher than before.

More formally, the proof is by induction on n. Assume that the existence of the desired word $M(w)$ is proven for any initial word w of length $|w| \le n$. Let us now prove it for the word $w_0 \dots w_n$ of length $n + 1$.

Denote $w = w_0 \dots w_{n-1}$. If the symbol w_n is not contained in $M(w)$, then the word $w' := M(w)w_n$ is the desired one. Otherwise consider the last occurrence of w_n in the word $M(w)$: let $M(w) = uw_nv$, where the symbol w_n does not appear in the word v. Compare the images of the initial point under the action of the words uw_n and $uw_nvw_n = M(w)w_n$ (both these images belong to the interval I_{w_n}). If the second of these images lies below the first one, the conclusion of the lemma is satisfied for the word $w' := M(w)w_n$. Otherwise one can take $w' := uw_n$. □

Recall that the original sequence of iterations of the point (k_0, x_0) is dense in $\operatorname{supp} \mu$. Combining this with the previous proposition, we obtain the following

Proposition 6.6. *There exists a sequence of words* w^j*, starting with the symbol* k_0 *and ending with* k*, such that for any of these words the corresponding iterations of the point* (k_0, x_0) *are monotonous downwards, and the final images* (k, x_j) *tend to* $(k, A_{\mu,k})$ *as* $j \to \infty$*. Also, the sequence of lengths* $|w^j|$ *tends to infinity.*

Proof. First, take a subsequence of iterations of (k_0, x_0) that tends to $(k, A_{\mu,k})$. Then apply Prop. 6.5 to obtain a sequence of words w_j such that the sequence (k, x_j) is

monotone. Because $I_{\mu,k}$ is invariant, for every j we have $A_{\mu,k} \leq x_j \leq B_{\mu,k}$. By the sandwich rule, $(k, x_j) \to (k, A_{\mu,k})$ as $j \to +\infty$.

Now assume the lengths $|w^j|$ do not tend to infinity. Then there exists an admissible composition G that takes the initial point (k_0, x_0) exactly to the point $(k, A_{\mu,k})$. But this is impossible. Indeed, otherwise the image of the point $P = (k_0, A_{\mu,k_0})$ under G must be below $(k, A_{\mu,k})$, because the point P is below (k_0, x_0). And at the same time $G(P) \in \operatorname{supp} \mu \subset [A_{\mu,k}, B_{\mu,k}]$. $\square$

Now we can conclude the proof of Lemma 6.3. Indeed, consider the sequence of words (w^j) constructed in Prop. 6.6. As their lengths tend to infinity, at some point they exceed N; in particular, there are repeating symbols in these words.

Let $|w^j| > N$. Passing from the end of this word to the beginning, find the first repetition of symbols. Namely, let

$$w^j = asbsc,$$

where a, b, c are words, s is a symbol, and all the symbols in the word bsc are different.

So the composition f_{sb} is a simple return. It takes the point $f_a(x_0)$ to a point below $f_a(x_0)$. Meanwhile, both these points belong to $\operatorname{supp} \mu$. Thus for any $M \geq 0$ the point $f_{sb}^M(f_a(x_0))$ also belongs to $\operatorname{supp} \mu$. The sequence $f_{sb}^M(f_a(x_0))$ tends to an attracting fixed point p^j of the map f_{sb} as $M \to +\infty$. Hence $p^j \in \operatorname{supp} \mu$.

Denote by g^j the simple transition f_{sc}. Note that $f_{sc}(p^j) \leq f_{asbsc}(x_0)$ because $p^j \leq f_{asb}(x_0)$. So the sequence $f_{w^j}(x_0)$ majorizes the sequence $g^j(p^j)$. Hence the latter also converges to $(k, A_{\mu,k})$.

But there is only a finite number of simple returns and of simple transitions. By the genericity conditions any simple return has finitely many fixed points. So there are only finitely many different maps g^j and points p^j. The sequence we have just constructed ranges over a finite set. The fact this sequence converges to $(k, A_{\mu,k})$ implies that the point $(k, A_{\mu,k})$ belongs to this finite set, so $(k, A_{\mu,k}) = g^j(p^j)$ for some j.

The same reasoning proves the statement for $B_{\mu,k}$. $\square$

Proof of Lemma 6.2. First, make an observation that will be useful later in the proof. Let G be any simple return. Then either $A_{\mu,k}$ is a fixed point of G or $G(A_{\mu,k}) \in (A_{\mu,k}, B_{\mu,k})$. In the first case, $A_{\mu,k}$ is hyperbolic by genericity condition i, and it is attracting by Lemma 6.3 and genericity condition ii. The same holds for the point $B_{\mu,k}$.

Denote by $\Phi(D)$ the *diffusion* of a set $D \subset \mathcal{I}$:

$$\Phi(D) = \bigsqcup_m \left(\bigcup_{k\colon (k,m)\text{ admissible}} f_k(D \cap I_k) \right),$$

$k, m \in \{1, \dots, N\}$. Note that a domain D is non-strictly trapping if and only if $\Phi(D) \subset D$, and trapping iff $\Phi(D) \subset \operatorname{int} D$. (Note also that the domain D is nonstrictly expelling, that is, the strip corresponding to D by (4.3), is nonstrictly expelling, if and only if $\Phi(D) \supset D$. In particular, $\Phi(D)$ is not necessary a superset of D.)

To construct the desired trapping domain, we will first prove the following statement. For any $\varepsilon > 0$, consider closed ε-neighborhood $D = \bigsqcup_k \overline{U_\varepsilon(I_{\mu,k})}$ of the set $\mathcal{I}_\mu$. Then, for any sufficiently small $\varepsilon > 0$ the $N+1$-th image of D is inside the interior of the union of the first N images:

$$\Phi^{N+1}(D) \subset \operatorname{int} \bigcup_{j=0}^{N} \Phi^j(D). \tag{6.2}$$

Indeed, take any admissible word $w = w_1 \dots w_{N+1}$ of length $N+1$. We want to study the image $f_{w_1 \dots w_N} : I_{w_1} \to I_{w_{N+1}}$ of $[A_{\mu,w_1} - \varepsilon, B_{\mu,w_1} + \varepsilon]$; it suffices to show that both endpoints of this image belong to the interior of the right hand-side of (6.2). Also, it suffices to consider only the images of $A_{\mu,w_1} - \varepsilon$: for the second endpoint, the argument is analogous.

Consider first all the intermediate images of the non-shifted endpoint $f_{w_1 \dots w_j}(A_{\mu,w_1})$. Note, that if at least one of them belongs to the interior of the corresponding $I_{\mu,w_{j+1}}$, then so does the final image $f_{w_1 \dots w_N}(A_{\mu,w_1})$, and thus by continuity for any sufficiently small ε so does $f_{w_1 \dots w_N}(A_{\mu,w_1} - \varepsilon)$.

Otherwise, if all the images $f_{w_1 \dots w_j}(A_{\mu,w_1})$ exactly coincide with the corresponding endpoints $A_{\mu,w_{j+1}}$, note that in the word w there is at least one simple return $f_{w_i \dots w_{j-1}}$ (so that $w_i = w_j$). Let $G = f_{w_i \dots w_{j-1}} : I_{w_i} \to I_{w_i}$ be the corresponding return map. Then, the point A_{μ,w_i} is an attracting fixed point of G (due to the argument in the beginning of the proof), and we have $A_{\mu,w_i} = f_{w_1 \dots w_{i-1}}(A_{\mu,w_1}) = f_{w_1 \dots w_{j-1}}(A_{\mu,w_1})$. Hence, for any sufficiently small $\varepsilon > 0$,

$$f_{w_1 \dots w_{i-1}}(A_{\mu,w_1} - \varepsilon) < G \circ f_{w_1 \dots w_{i-1}}(A_{\mu,w_1} - \varepsilon) = f_{w_1 \dots w_{j-1}}(A_{\mu,w_1} - \varepsilon),$$

what implies

$$f_{w_1 \dots w_{i-1} w_j \dots w_N}(A_{\mu,w_1} - \varepsilon) < f_{w_1 \dots w_N}(A_{\mu,w_1} - \varepsilon).$$

The point $f_{w_1 \dots w_N}(A_{\mu,w_1} - \varepsilon)$ belongs to the interior of $\cup_{j=0}^N \Phi^j(D)$. So, for any admissible word w of length $N+1$ we have the desired inclusion, and (6.2) is proven.

Now, denote by Φ_δ the *δ-dispersed diffusion*:

$$\Phi_\delta(D) := \overline{U_\delta(\Phi(D))}.$$

For any sufficiently small $\varepsilon > 0$, as the inclusion (6.2) is stable under small perturbations, we have for any sufficiently small $\delta > 0$

$$\Phi_\delta^{N+1}(D) \subset \operatorname{int} \bigcup_{j=0}^{N} \Phi_\delta^j(D). \tag{6.3}$$

Consider then for any such ε, δ the domain $\tilde{D}_\delta := \bigcup_{j=0}^N \Phi_\delta^j(D)$. Immediately from definition we see that for any domain Y one has $\Phi(Y) \subset \operatorname{int} \Phi_\delta(Y)$, in particular,

$$\Phi(\Phi_\delta^j(D)) \subset \operatorname{int} \Phi_\delta^{j+1}(D), \quad j = 0, \dots N.$$

Hence,

$$\Phi(\tilde{D}_\delta) \subset \operatorname{int} \tilde{D},$$

and $\tilde{D}$ is a trapping domain. □

Any ergodic stationary measure μ is supported inside some non-strictly trapping domain that is a union of intervals $\mathcal{I}_\mu = \bigsqcup_k I_{\mu,k}$. The ends of these intervals are images of fixed points of simple returns under simple transitions. So there are only finitely many possibilities for such trapping domains. Now the conclusions i, ii of Thm. 4.7 are reduced to the following two lemmas.

Lemma 6.7. *For any two ergodic stationary measures μ_1 and μ_2 the corresponding intervals $I_{\mu_1,k}$ and $I_{\mu_2,k}$ are either disjoint for any k or coincide for any k. In the former case they are situated in the same order on all the intervals I_k.*

Lemma 6.8. *Any stationary ergodic measure μ is the unique stationary measure in the corresponding trapping domain $\mathcal{I}_\mu$.*

To prove them, we need the following (useful for many reasons) lemma and its corollaries.

Lemma 6.9. *Let μ be an ergodic stationary measure, k and k' be two arbitrary symbols. Then for any $\varepsilon > 0$ there exist admissible compositions $G_{A,B} \colon I_k \to I_{k'}$ such that*

$$G_A(I_{\mu,k}) \subset U_\varepsilon(A_{\mu,k'}), \quad G_B(I_{\mu,k}) \subset U_\varepsilon(B_{\mu,k'}).$$

Proof. Apply the genericity condition iii (sf. Sec. 5) to the set of lower endpoints $\{A_{\mu,k}\}$ of intervals $I_{\mu,k}$. There exist k'' and an admissible composition $G_1 \colon I_k \to I_{k''}$ such that $p := G_1(A_{\mu,k}) > A_{\mu,k''}$. Because $A_{\mu,k''} = \inf \operatorname{supp} \mu_{k''}$, $\mu([A_{\mu,k''}, p]) > 0$. Thus the generic start point (x_0, k'') from the proof of Lemma 6.3 can be chosen from $[A_{\mu,k''}, p]$.

Because μ is ergodic, a random orbit with a μ-generic start point is dense in $\operatorname{supp} \mu$ almost surely. So for any $\varepsilon > 0$ there exists an admissible composition $G_2 \colon I_{k''} \to I_{k'}$ such that $G_2(x_0, k'') \in U_\varepsilon(B_{\mu,k'})$.

The monotonicity of fiberwise maps f_m implies that the image $(G_2 \circ G_1)(I_{\mu,k})$ is situated on the interval $I_{k'}$ strictly above the point $G_2(x_0, k'')$. The map $G_B := G_2 \circ G_1$ is constructed. The procedure for the lower endpoint $A_{\mu,k'}$ is analogous. □

Corollary 6.10. *For any interval I_k there exist an interval I_m and admissible compositions $G_{1,2} \colon I_m \to I_k$ such that the images $G_1(I_{\mu,m})$ and $G_2(I_{\mu,m})$ do not intersect.*

Proof. Indeed, it suffices to take the compositions that send the interval I_k to disjoint neighborhoods of the points $A_{\mu,m}$ and $B_{\mu,m}$ respectively. □

Corollary 6.11. *For any interval I_k there exist an interval I_m and an admissible composition $G\colon I_m \to I_k$ such that $\mu_m \neq G_*\mu_k$.*

Proof of Lemma 6.7. Assume the converse: let the intervals $I_{\mu_1,k}$ and $I_{\mu_2,k}$ intersect but not coincide. Then there is an endpoint of one of them that does not belong to another; without loss of generality let it be the point $A_{\mu_1,k}$. Take a neighborhood $U \ni A_{\mu_1,k}$ such that $I_{\mu_2,k} \cap U = \emptyset$.

By Lemma 6.9 applied for $k = k'$, there exists an admissible return G such that $G(I_{\mu_1,k}) \subset U$. On the other hand, any interval $I_{\mu,k}$ is absorbing under any admissible return, so $G(I_{\mu_2,k}) \subset I_{\mu_2,k}$. Then the nonempty set $G(I_{\mu_1,k} \cap I_{\mu_2,k})$ is contained in both U and $I_{\mu_2,k}$. This contradiction proves the first part of the lemma.

The second part of the lemma follows directly from the monotonicity of the fiberwise maps. □

As the diffeomorphisms f_ω take the interval I to its interior, the inverse maps f_ω^{-1} are not everywhere defined. To overcome this technical obstacle, we extend the interval fibers I to circle fibers $S^1 \supset I$ in the following way:

- each map f_ω is an orientation-preserving diffeomorphism of the circle;
- on the set $S^1 \setminus I$ each map f_ω is strictly stretching: $f'_\omega > 1$;
- the whole skew product is generic in the sense of Sec. 5.

Then for the inverse skew product there exists a single attractor J in the strip $\Sigma \times (S^1 \setminus I)$. Moreover, it is a (non-bony) graph of a continuous function $j\colon \Sigma \to S^1$. This statement is straightforward. The detailed proof is given, for instance, in [16, Prop. 2,3].

For any $n \in \mathbb{Z}$ denote by $f_{n,\omega}(x)$ the nth iterate of a fiber point x with a sequence ω in the base, so

$$F^n(\omega, x) = (\sigma^n\omega, f_{n,\omega}(x)). \tag{6.4}$$

Then for any $n > 0$

$$f_{n,\omega} = f_{\sigma^{n-1}\omega} \circ f_{\sigma^{n-2}\omega} \circ \ldots \circ f_\omega, \quad \text{and} \quad f_{-n,\omega} = f^{-1}_{\sigma^{-n}\omega} \circ f^{-1}_{\sigma^{-n+1}\omega} \circ \ldots \circ f^{-1}_{\sigma^{-1}\omega} = f^{-1}_{n,\sigma^{-n}\omega}.$$

In particular, for step skew products

$$f_{n,\omega} = f_{\omega_{n-1}} \circ f_{\omega_{n-2}} \circ \ldots \circ f_{\omega_0}, \quad \text{and} \quad f_{-n,\omega} = f^{-1}_{\omega_{-n}} \circ f^{-1}_{\omega_{-n+1}} \circ \ldots \circ f^{-1}_{\omega_{-1}}.$$

Suppose μ is an ergodic stationary measure of $\Pi(F)$. By Lemma 6.2, the non-strictly trapping domain $\mathcal{I}_\mu$ has arbitrary small strictly trapping neighborhood $U(\mathcal{I}_\mu) = \sqcup_k U_k$. Denote by $\tilde{D} \subset \Sigma \times S^1$ the corresponding strip in the two-sided skew product:

$$\tilde{D} := \bigsqcup_k C_k \times U_k,$$

where $C_k = \{\,\omega \mid \omega_0 = k\} \subset \Sigma$. By Prop. 4.5, $\tilde{D}$ is also strictly trapping.

Proposition 6.12. *The maximal attractor $A_{\max}$ of the trapping strip $\tilde{D}$ is an invariant continuous-bony graph.*

Proof. Like before, we denote by A_ω the intersection of $A_{\max}$ and the fiber $\{\omega\} \times I$. By definition,

$$A_\omega = \bigcap_{n \geq 0} A_\omega^{(n)},$$

where

$$A_\omega^{(n)} = f_{-1} \circ \cdots \circ f_{\omega_{-n}}(\tilde{D}_{\sigma^{-n}\omega}).$$

Note that $A_\omega^{(n)}$ is a sequence of nested intervals, and thus A_ω is either an interval or a single point. Also note that if some sequences ω and ω' are close enough to each other, say,

$$\omega'_{-n} = \omega_{-n}, \ldots, \omega'_{-1} = \omega_{-1},$$

then $A_\omega^{(n)} = A_{\omega'}^{(n)} \supset A_{\omega'}$. This implies the upper-semicontinuity of A_ω and thus the continuity of $A_{\max}$, see Def. 2.6.

To prove that $A_{\max}$ is actually a bony graph, we use an argument similar to the [22, Theorem 3] by Kudryashov. By the Fubini Theorem, $A_{\max}$ is a bony graph iff its standard measure $\mathbf{s}$ is zero. For every $k = 1 \ldots N$ and $x \in I_{\mu,k}$, denote by $\Omega_{k,x} \subset \Sigma$ the slice of $A_{\max} \cap \{\omega_0 = k\}$ by the horizontal line $\Sigma \times \{x\}$. Because $F^{-1}(A_{\max}) = A_{\max}$, $\Omega_{k,x}$ is the set of sequences such that

- $\omega_0 = k$;
- $\forall n \geq 0 \quad F^{-n}(\omega, x) \in \tilde{D}$.

By definition, $A_{\max} = \sqcup_{k,x} \Omega_{k,x} \times \{x\}$.

Now we show that for all k, for all $x \in I_{\mu,k}$ we have $\nu(\Omega_{k,x}) = 0$. By the Fubini Theorem, this will imply $\mathbf{s}(A_{\max}) = 0$. In a sense, we have just switched the order of integration for $A_{\max}$.

To prove this, we cover $\Omega_{k,x}$ by a disjoint union of cylinders of arbitrary small ν-measure.

1st generation of the cylinders. By Cor. 6.10 of Lemma 6.9, there exist two words $w^{(1)}, w^{(2)}$ such that for all $i = 1, 2$

- the word $w^{(i)}\omega_0$ is admissible;
- $w_1^{(i)} = \omega_0$;
- $|w^{(1)}| = |w^{(2)}| =: L$;
- $f_{w^{(1)}}(I_{\mu,k}) \cap f_{w^{(2)}}(I_{\mu,k}) = \emptyset$.

The latter implies that for at least one of these words, denote it by w, holds $f_w^{-1}(x) \notin I_{\mu,k}$. Now for every $\omega \in \Omega_{k,x}$ we know that $\omega_{[-L,-1]} \neq w$, so

$$\Omega_{k,x} \subset \bigsqcup_{s \neq w} \left\{ \omega_{[-L,-1]} = s \right\}.$$

The right-hand side is the 1st generation of the cylinders.

The next generation of the cylinders. Now we subdivide each of the 1st generation cylinders and see that

$$\left\{ \omega_{[-L,-1]} = s \right\} = \bigsqcup_{s'} \left\{ \omega_{[-2L,-1]} = s's \right\}.$$

For each $s \neq w$, we now have new $x_s := f_s^{-1}(x) \in I_{\mu,k}$ and a new word $w' \in \{w^{(1)}, w^{(2)}\}$ such that $f_{w'}^{-1}(x_s) \notin I_{\mu,k}$. Thus

$$\Omega_{k,x} \cap \left\{ \omega_{[-L,-1]} = s \right\} \subset \bigsqcup_{s' \neq w'} \left\{ \omega_{[-2L,-1]} = s's \right\}. \tag{6.5}$$

Uniform decrease of the cylinders. Let w be any admissible word. For any cylinder $C = \left\{ \omega_{[n_1,n_2]} = u \right\}$, u being some word, denote by wC the cylinder

$$wC := \left\{ \omega_{[n_1-L,n_2]} = wu \right\},$$

provided wu is also admissible. In other words, we concatenate w and u. Obviously, $wC \subset C$.

Proposition 6.13. *There exists $0 < \lambda < 1$ such that for any $w \in \{w^{(1)}, w^{(2)}\}$ and for any cylinder C such that wu is admissible, we have*

$$\frac{\nu(wC)}{\nu(C)} > \lambda.$$

Because ν is Markov, the Proposition is obviously true. We do not know what exactly is the class of shift-invariant measures with full support that possess this property. We suspect this holds for all the measures which appear as SRB in smooth partially hyperbolic dynamics, and perhaps for all or a large class of Gibbs measures. The question seems to be interesting on its own. Once one could prove Prop. 6.13 for a larger class of measures, this would immediately generalize our Thm. 2.15 to them.

End of the proof of Prop. 6.12. On each step of the construction, we subdivide the cylinders in the same way. Because of Prop. 6.13, each next generation of cylinders has at least λ part of the "bad" ones, i.e. those who eventually throw x out of $\tilde{D}$. So each subdivision reduces the ν-measure of the cover by at least $1 - \lambda < 1$ times. So we are able to cover $\Omega_{k,x}$ with a set of an arbitrary small ν-measure. Thus $\Omega_{k,x} = 0$, and Prop. 6.13 is proved.

□

Proposition 6.14. *The lift $\mathbf{m}_\varphi$ of the measure ν on the graph part Γ_φ of $A_{\max}$ is SRB.*

Proof. Assume that the point $(\omega, x) \in S_{\varphi_1,\varphi_2}$ is such that the sequence of the time averages of its base coordinate $\omega \in \Sigma$ converges to the measure ν. (Because the measure ν is ergodic, this holds for the set of points of full standard measure in S_{φ_1,φ_2}.)

Consider the sequence of time averages

$$\theta_n := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{F^j(\omega,x)}.$$

We have to show that this sequence converges (in $*$-weak topology) to the measure $\mathbf{m}_\varphi$. To do so, note, on one hand, that any limit point $\theta = \lim_i \theta_{n_i}$ of this sequence of measures due to the choice of the coordinate ω is projected to the measure ν on the base. Moreover, its support is contained in the maximal attractor $A_{\max} := \cap_j F^j(S_{\varphi_1,\varphi_2})$. On the other hand, ν-almost every fiber $h^{-1}(\omega')$ intersects the maximal attractor by a single point $(\omega', \varphi(\omega'))$. Thus the conditional measure of θ on almost every fiber is the Dirac measure $\delta_{(\omega',\varphi(\omega'))}$, and hence the measure θ itself coincides with the measure $\mathbf{m}_\varphi$.

The uniqueness of the limit point and compactness of the space of measures now imply that all the sequence of time averages θ_n converges to the measure $\mathbf{m}_\varphi$. □

Remark 6.15. The same arguments imply than for any measure $\mathbf{m}$, supported in S_{φ_1,φ_2}, that projects to the Markov measure ν on the base, its iterations converge to the measure $\mathbf{m}_\varphi$. Indeed, any accumulation point of the sequence of measures $F_*^n\mathbf{m}$ is a measure, supported on the maximal attractor A_{max} and such that it projects to the Markov measure on the base. Hence, any accumulation point of this sequence of iterations coincides with $\mathbf{m}_\varphi$ and thus the entire sequence converges to $\mathbf{m}_\varphi$.

Proof of Lemma 6.8. As we have already mentioned in Sec. 4, in a skew product over the Markov shift in the space of one-sided sequences Σ_+ to the measure μ' corresponds measure $\mathbf{m}'_+$, defined by formula (4.2). As one can easily see from the definition, $\mathbf{m}'_+$ is F_+-invariant.

Consider now the measure $\mathbf{m}' = \sum_k \nu_k \times \mu_k$, that is given by the same sum, but for the skew product over the shift on space of two-sided sequences. In general, $F_*\mathbf{m}' \neq \mathbf{m}'$. However, as the measure $\mathbf{m}'_+$ is F_+-invariant, all the iterations $F_*^n\mathbf{m}'$ project to the measure $\mathbf{m}'_+$ on the skew product over the shift on space of one-sided sequences. Hence, projection on $\mathcal{I}$ of any of these iterations gives us the measure μ' (which is the projection of $\mathbf{m}'_+$).

By construction, the measure $\mathbf{m}'$ projects to the measure ν on the base. Due to Remark 6.15, the iterations $F_*^n\mathbf{m}'$ of the measure $\mathbf{m}'$ converge to the measure $\mathbf{m}_\varphi$. On the other hand, all these iterations project to the measure μ on $\mathcal{I}$, thus the same holds for their limit. Hence, the measure μ is the projection on $\mathcal{I}$ of the measure $\mathbf{m}_\varphi$. This projection is the distribution $\varphi_*\nu$ of the values of the map φ, that corresponds to the graph part of the maximal attractor. As we supposed only that the measure μ' is a

stationary measure on the domain $\mathcal{I}_\mu$, the stationary measure on this domain is unique (and hence it is the measure μ).

□

7. Smoothing Stochastic Perturbations and Baxendale's Theorem

The goal of this Section is to complete the proof of Thm. 4.7. The conclusions i and ii of the theorem follow from Lemmata 6.3, 6.7 and 6.8. To prove that the (random) Lyapunov exponent for stationary measure is negative, we follow the strategy of a theorem by Baxendale (sf. [4, Theorem 4.2]). For this, we introduce stochastic skew products over discrete Markov chains and prove for them an analogue of the Baxendale's theorem.

7.1. Stochastic Skew Products. A stochastic skew product over a Markov chain is an object of 3 components:

i) a discrete Markov chain with N states;
ii) a manifold M;
iii) N random variables ξ_{ij} taking the values in $\mathrm{Diff}^r(M)$, $r \geq 0$.

Recall the Markov process $\Pi(F)$ defined in Section 4. Fix any ergodic stationary measure μ of $\Pi(F)$. Restrict $\Pi(F)$ to the trapping region $\mathcal{I}_\mu$. The process $\Pi(F)$ can be viewed as a stochastic skew product over a Markov chain with

i) the Markov chain equal to our given Markov chain in the base;
ii) $M = \mathcal{I}_\mu$;
iii) ξ_{ij} equal to the δ-measures at $f_i \in \mathrm{Diff}^1(M)$.

Now we introduce the smoothed version of $\Pi(F)$. Let ζ be any random variable with a C^∞-density supported on $[-1, 1]$. For $0 < \varepsilon < 1$, denote $\zeta^\varepsilon := \varepsilon \cdot \zeta$. Denote by $\Pi^\varepsilon(F)$ the stochastic skew product with $\xi^\varepsilon_{ij} = \xi_{ij} * \zeta^\varepsilon$. Every stationary measure of $\Pi^\varepsilon(F)$ has a smooth density.

7.2. Relative Entropy. For any measure μ let $\mu_k = \mu|_{I_k}$, $k = 1, \dots, N$, and assume they are normalized so that $\mu_k(I_k) = 1$. Its volume Lyapunov exponent is given by the integral

$$\lambda_{vol} = \sum_{i,j} p_i \pi_{ij} \cdot \mathbb{E} \int_M \log \mathrm{Jac}\, \xi_{ij}|_x d\mu_i(x) \tag{7.1}$$

The same argument as in [4, Theorem 4.2] tells us that if the measures μ_i are absolutely continuous w.r.t. Lebesgue, (7.1) can be rewritten in terms of so-called *relative entropy*. Namely, for any two measures m_1, m_2 define

$$h(m_1|m_2) := \sup_{\psi \in C(M)} \left[\log \int e^\psi \, dm_1 - \int \psi \, dm_2 \right] = \int \left(\log \frac{dm_1}{dm_2} \right) dm_1$$

if $m_1 \ll m_2$, and $h(m_1|m_2) := \infty$ otherwise. Then

$$\lambda_{vol} = -\sum p_i \pi_{ij} \cdot \mathbb{E} h((\xi_{ij})_* \mu_i | \mu_j). \tag{7.2}$$

Indeed, the difference between (7.1) and (7.2) is of zero integral (what is easy to check noticing that

$$\log \frac{dm_1}{dm_2} = \log \rho_1 - \log \rho_2$$

provided the measures m_1, m_2 are both absolutely continuous w.r.t. Lebesgue with the densities ρ_1, ρ_2.

7.3. Baxendale's Theorem. For any $\varepsilon > 0$ for the smoothed $\Pi^\varepsilon(F)$ there exists at least one ergodic Π_ε-stationary measure μ_ε. For μ_ε we have

$$\lambda^\varepsilon_{vol} = -\sum p_i \pi_{ij} \cdot \mathbb{E} h((\xi^\varepsilon_{ij})_* \mu^\varepsilon_i | \mu^\varepsilon_j) < 0.$$

Because the space of measures on M is compact, we can produce a sequence $\varepsilon_n \to 0$ such that $\mu_i^{\varepsilon_n}$ weakly converges to some measure μ_i. Then either $\lambda^{\varepsilon_n}_{vol} \to 0$ and thus $h((\xi^{\varepsilon_n}_{ij})_* \mu_i | \mu_j) \to 0$ weakly, or there exists a subsequence $\varepsilon'_n \to 0$ such that $\lambda^{\varepsilon'_n}_{vol} \to -\alpha < 0$.

Note that the relative entropy is semicontinuous as a supremum of continuous functionals. Hence, in the former case any admissible (i, j) one has almost surely $h((\xi_{ij})_* \mu_i | \mu_j) = 0$, and thus $(\xi_{ij})_* \mu_i = \mu_j$. But this is impossible by Corollary 6.11.

In the latter case

$$\sum_{i,j} p_i \pi_{ij} \int \log \operatorname{Jac} \xi_{ij}(x)\, d\mu_i(x) = \lim_{n\to\infty} \sum_{i,j} p_i \pi_{ij} \int \log \operatorname{Jac} \xi^\varepsilon_{ij}(x)\, d\mu^\varepsilon_i(x) = -\alpha < 0,$$

and hence for at least one ergodic component of μ we have $\lambda_{vol}(\mu) \le -\alpha < 0$. But the stationary measure in this domain is unique (Lemma 6.8), and hence its Lyapunov exponent is negative.

The proof of Thm. 4.7 is complete.

8. Return to step skew products: proof of the main theorem

Proposition 8.1. *Let the map $\varphi(\omega)$, $\varphi\colon \Sigma \to S^1$, depend only on finitely many symbols in ω. Suppose its graph Γ moves up (down). Then*

i) *the pointwise limit of its iterates $\Gamma_n = F^n(\Gamma)$ as $n \to +\infty$ is the graph of a measurable function $\varphi_{+\infty}\colon \Sigma \to S^1$;*
ii) *the function $\varphi_{+\infty}(\omega)$ does not depend on "future" of ω:*
$$\forall i \in \mathbb{Z},\ \forall \omega = \omega_{[-\infty,-1]}\omega_{[0,+\infty]}, \omega' = \omega_{[-\infty,-1]}\omega'_{[0,+\infty]}, \quad \varphi_{+\infty}(\omega) = \varphi_{+\infty}(\omega');$$
iii) *$\varphi_{+\infty}$ is invariant under F:*
$$\varphi_{+\infty}(\sigma\omega) = f_\omega(\varphi_{+\infty}(\omega)).$$

Remark 8.2. The analogous statement holds for $n \to -\infty$. The limit function $\varphi_{-\infty}$ does not depend on the "past" of ω.

Proof. Take any $\omega \in \Sigma$. Let φ_n be the function that corresponds to the graph Γ_n. Because Γ moves up (down), the sequence $(\varphi_n(\omega))$ is monotone. It is also bounded because of the invariant repeller graph $J \subset \Sigma \times (S^1 \setminus I)$. Thus $\varphi_{+\infty}$ is well defined. Because $\varphi_{+\infty}$ is a pointwise limit of continuous functions φ_n, it must be measurable.

Now let $\varphi(\omega)$ depend only on $\omega_{-k}, \dots, \omega_m$ in ω. The function φ_n has the form

$$\varphi_n(\omega) = f_{n,\sigma^{-n}\omega}(\varphi(\sigma^{-n}\omega)).$$

Then for every $n \geq m+1$ the function φ_n depends only on $\omega_{-k-n}, \dots, \omega_{-1}$. In particular, φ_n does not depend on $\omega_{[0,+\infty]}$. Thus the limit function $\varphi_{+\infty} = \lim_{n\to+\infty} \varphi_n$ is also independent of $\omega_{[0,+\infty]}$.

The invariance of $\varphi_{+\infty}$ immediately follows from the definition of limit. $\square$

For any measurable function φ denote by $\mathbf{m}_\varphi$ the lift of the base measure ν to the graph Γ of φ. The measure $\mathbf{m}_\varphi$ is invariant provided that Γ is invariant.

Proposition 8.3. *Suppose a function $\varphi \colon \Sigma \to S^1$ is independent of the future and the graph of φ is F-invariant. Then there exists a stationary measure μ of the process $\Pi(F)$ such that $\pi_*\mathbf{m}_\varphi = \mathbf{m}_+(\mu)$, where measure $\mathbf{m}_+(\mu)$ is defined by Eq. (4.2).*

Proof. Because φ is independent of the future, the projection of the measure $\mathbf{m}_\varphi$ to $\Sigma_+ \times S^1$ has the following simple form. The restriction of the projection to each cylinder $\mathcal{C}_k$ it is the Cartesian product of ν_k^+ and some measure μ_k on I_k, see (4.2). These measures μ_k constitute the desired stationary measure μ on $\mathcal{I}$. $\square$

Remark 8.4. By definition, the fiberwise Lyapunov exponent of $\mathbf{m}_\varphi$ equals to the random Lyapunov exponent of μ.

Remark 8.5. The analogous statements holds for any φ that is independent of the past. Measure μ is stationary for the process $\Pi(F^{-1})$.

Now we are ready to prove Thm. 2.15. By Thm. 4.7, the process $\Pi(F)$ has finitely many ergodic stationary measures. Their supports are contained in disjoint vertically sorted trapping regions. These regions correspond to the trapping strips in $\Sigma \times S^1$. By Prop. 6.12, the maximal attractors of these strips are CBGs. Thus any strip has a unique invariant measure projecting to ν in the base. By Prop. 6.14, this measure is SRB. The claims ii, iii are established. Prop. 2.7 implies claim iv.

Because the fiber maps are monotonous, the complement to the disjoint union of the trapping strips is also the disjoint union of finitely many vertically ordered step strips. The F-images of these strips are step w.r.t. the symbol ω_{-1} of ω. These new strips are inverse trapping.

Let us see what happens in any of these strips. According to Prop. 8.1, Prop. 8.3, the limits of the backward iterates of its lower and upper boundaries are some invariant measurable graphs Γ_L and Γ_U. They support invariant measures, namely, the lifts of ν. These invariant measures in turn correspond to some stationary measures $\mathbf{m}_L$, $\mathbf{m}_U$ in $\Pi(F^{-1})$. Suppose $\mathbf{m}_L$ and $\mathbf{m}_U$ do not coincide. Then there is an invariant attracting graph between them which has its own trapping strip. Thus the inverse trapping strip can decomposed into at least two strips. This contradiction proves that $\mathbf{m}_L = \mathbf{m}_U$. The unique stationary measure $\mathbf{m}_L = \mathbf{m}_U$ corresponds to the unique repeller within the inverse trapping strip. The claim vi is proven.

The claim v follows from Rem. 8.4.

Finally, the moving up (down) border of any trapping strip converges a.e. to the corresponding attractor as $n \to +\infty$ and to the corresponding repeller as $n \to -\infty$. This makes a.e. point between the attractor and the repeller do the same. Thus the basins of attractors and repellers cover the whole phase space $\Sigma \times S^1$ with the exception of zero measure set in the base. The claim i is proven. Thm. 2.15 is complete.

We conclude this paper with an estimate of the number of attractors and repellers. For any f, denote by $\mathrm{AFix}(f)$ the set of all attracting fixed points of f.

Proposition 8.6. *Let $\omega \in \Sigma$ be any periodic sequence, $\omega = \dots www \dots$, where w is a simple return. Let $\tilde{D}$ be any trapping strip. Then there exist $a_1, a_2 \in \mathrm{AFix}(f_w)$ (perhaps, $a_1 = a_2$) such that $A_{\max}(\tilde{D}) \cap I_\omega = [a_1, a_2]$.*

Proof. Indeed, because $\tilde{D}$ is trapping, $f_w(\tilde{D}_\omega) \subset \tilde{D}_\omega$, and $A_{\max}(\tilde{D}) \cap I_\omega = \bigcap_{n \geq 0} f_w^n(\tilde{D}_\omega)$ which has to be an interval between some attracting fixed points of f_w. □

Corollary 8.7. *The number of the attracting CBGs is less or equal to* $\max_w \sharp\, \mathrm{AFix}(f_w)$, *where w runs over all possible simple returns.*

Remark 8.8. This upper bound is sharp.

Remark 8.9. For the full shift in the base, the bound is just $\max_i \sharp\, \mathrm{AFix}(f_i)$.

Remark 8.10. The analogous statements hold for the repelling CBGs.

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